MATH 312
Lecture 20

LINEAR DIFFERENTIAL SYSTEMS

Recap:

- The EIGENVALUES of an $n \times n$ matrix A are solutions λ of $\det(A - \lambda I) = 0$. ← degree n polynomial
- For each eigenvalue $\overline{\lambda}$ of A, the collection of EIGENVECTORS is given by the nonzero vectors in $N(A - \overline{\lambda} I)$.
- If the eigenvalues $\lambda_1, \dots, \lambda_n$ (possibly repeated!) of A have n independent eigenvectors $\vec{v}_1, \dots, \vec{v}_n$, then A is DIAGONALIZABLE
- That is to say,

$$A = SDS^{-1},$$

where $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ is diagonal,

and $S = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$ is invertible.

- Upshot: ability to compute $A^k = SD^kS^{-1}$.

Notes: If A is diagonalizable, then

A) $$\det(A) = \det(SDS^{-1}) = \cancel{\det(S)} \det(D) \frac{1}{\cancel{\det(S)}}$$
$$= \det(D) = \underbrace{\lambda_1 \cdots \lambda_n}_{\text{product of eigenvalues!}}$$

B) <u>But</u> <u>be</u> <u>careful</u>!

(i) Row operations do <u>NOT</u> preserve eigenvalues.

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$\lambda_1 = 0$, $\lambda_2 = 5$ | $\lambda_1 = 0$, $\lambda_2 = 1$.

This is Important

(ii) So, $\begin{pmatrix} \text{product} \\ \text{of} \\ \text{pivots} \end{pmatrix} = \det(A) = \begin{pmatrix} \text{product} \\ \text{of} \\ \text{eigenvalues} \end{pmatrix}$

Even though there may be no direct relation between pivots & eigenvalues!

DIFFERENTIAL EQUATIONS (Linear)

Maybe the easiest differential equations in 1-d take on the form $\boxed{\dfrac{dx}{dt} = \alpha x}$ along with an initial condition,

$$x(0) = x_0.$$

ONE DIMENSIONAL

The solution only requires integration:

$$\int_0^t dx/x = \int_0^t \alpha \, dt,$$

so $\ln x(t) - \ln x(0) = \alpha t$

so $\ln x(t) = \alpha t + \ln x_0$

$$\text{so} \quad x(t) = e^{\alpha t + \ln x_0}$$

$$= e^{\alpha t} \cdot e^{\ln x_0}$$

$$\Rightarrow \boxed{x(t) = e^{\alpha t} x_0}$$

This solves the diff'l equation

$$\boxed{\begin{array}{l} dx/dt = \alpha x \\ x(0) = x_0 \end{array}}$$

COMPLETELY!

(Where $e^{y} = 1 + y + y^2/2! + \cdots = \sum_{j=0}^{\infty} y^j/j!$)

A standard (but silly) model for population growth is given by such an equation:

$$\underbrace{p'}_{\substack{\text{rate of change}\\ \text{of population}\\ \text{at time } t}} = \underbrace{\alpha}_{\substack{\text{fraction of}\\ \text{reproducing}\\ \text{"adults"}\\ 0 \leq \alpha \leq 1}} \underbrace{p}_{\substack{\text{current}\\ \text{population}\\ \text{at time}\\ t}}$$

" The more people there are, the more babies they will make ".

If the population at $t=0$ is $p(0)$, then the population at time t is given by

$$p(t) = e^{\alpha t} p(0) \leftarrow \text{BAM! exponential growth.}$$

A slightly more reasonable model of growth takes into account the natural resources that are available to the population at any given time.

Letting $r(t)$ be the resources at time t, we make the following assumptions:

$p'(t)$ is proportional to $p(t)$ AND $r(t)$ (+vely)
$r'(t)$ is proportional to $p(t)$ (−vely)

The first assumption says we need people AND resources to make more people. The second says that the more people we've got, the faster the resources will deplete. So:

Two linear differential equations

"coupled" together.

$$\begin{cases} p'(t) = \alpha p(t) + \beta r(t) \\ r'(t) = -\gamma p(t) \end{cases}$$

for α, β, γ positive constants

In matrix form: $\dfrac{d}{dt}\begin{bmatrix} p(t) \\ r(t) \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ -\gamma & 0 \end{bmatrix} \begin{bmatrix} p(t) \\ r(t) \end{bmatrix}$

Def A system $\vec{x}' = A\vec{x}$ where A is $n \times n$ is called a LINEAR DIFFERENTIAL SYSTEM (of dimension n).

When $n = 1$, we are back to $x' = \alpha x$ and know the solution given any initial $x(0) = x_0$. So:

Q How to solve the n-dim'l system given an initial vector $\vec{x}(0)$?

Note If $\vec{x}' = D\vec{x}$ where D is DIAGONAL, say $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ then there is no trouble: the n-dimensional system just decomposes into n one-dimensional linear differential equations:

$$\begin{cases} x_1'(t) = \lambda_1 x_1(t) \\ x_2'(t) = \lambda_2 x_2(t) \\ \vdots \qquad \vdots \qquad \vdots \\ x_n'(t) = \lambda_n x_n(t) \end{cases}$$

which are solved by using

$$x_j(t) = e^{\lambda_j t} \cdot x_j(0)$$

But wait: this means

$$\vec{x}(t) = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix} \vec{x}(0).$$

(This looks a LOT like $x(t) = e^{\lambda t} x(0)$, the 1D solution)

Def "matrix" e

The EXPONENTIAL e^A of an $n \times n$ matrix A is defined by the series

$$\boxed{\begin{aligned} e^A &= \mathrm{Id} + A + A^2/2! + \cdots \\ &= \sum_{j=0}^{\infty} A^j/j! \end{aligned}}$$

When $A = D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$, then we just have

$$e^D = \begin{bmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{bmatrix}.$$

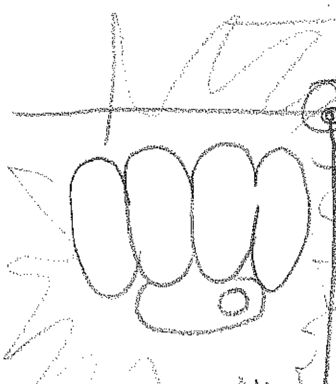
If A is NOT diagonal, then e^A looks like a MESS to compute. BUT this is what eigenvalues and eigenvectors are good for! IF A is diagonalIZABLE, then $A = SDS^{-1}$, and

$$e^A = \sum_{j=0}^{\infty} A^j/j! = \sum_{j=0}^{\infty} (SDS^{-1})^j/j!$$

$$= \sum_{j=0}^{\infty} SD^jS^{-1}/j! = S\left(\sum_{j=0}^{\infty} D^j/j!\right)S^{-1}$$

$$= \underline{\underline{Se^DS^{-1}}}$$

So, $e^A = S\begin{bmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{bmatrix}S^{-1}$, where λ's are eigenvalues of A, and S is the corresponding eigenvector matrix.



The n-dimensional system $\vec{x}' = A\vec{x}$ with initial conditions $\vec{x}(0)$ is solved by

$$\vec{x}(t) = S\begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix}S^{-1}\vec{x}(0)$$

(eigenvals)

PUNCHLINE!! whenever $A = SDS^{-1}$ is diagonalizable; $D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

Okay, enough theory: Example Time!

Eg Solve the 2D linear differential system
$$\vec{x}' = \begin{bmatrix} 7 & 5 \\ -10 & -8 \end{bmatrix} \vec{x},$$
with initial conditions $\vec{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

STEP 1 Compute eigenvalues of $A = \begin{bmatrix} 7 & 5 \\ -10 & -8 \end{bmatrix}$.

$$A - \lambda I = \begin{bmatrix} (7-\lambda) & 5 \\ -10 & (-8-\lambda) \end{bmatrix}$$

So $\det(A - \lambda I) = 0$ means

$$(7-\lambda)(-8-\lambda) + 50 = 0$$

$$\Rightarrow \lambda^2 + \lambda - 6 = 0 \Rightarrow (\lambda-2)(\lambda+3) = 0$$

So, $\underline{\lambda_1 = 2}$ and $\underline{\lambda_2 = -3}$

STEP 2 Compute eigenvectors for each λ.

$\lambda_1 = 2$, $A - 2I = \begin{bmatrix} 5 & 5 \\ -10 & -10 \end{bmatrix}$, $\vec{v_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$\lambda_2 = -3$, $A + 3I = \begin{bmatrix} 10 & 5 \\ -10 & -5 \end{bmatrix}$, $\vec{v_2} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

STEP 3 Write $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, $S = \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix}$ and compute S^{-1}.

$D = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$, $S = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$, $S^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.

STEP 4 Ans $= S e^{Dt} S^{-1} \vec{x}(0)$

$$= \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \ldots$$